

M.V.
M.W.C. 92, 94
Q.No. 96, 92

Let T be a normal operator on a finite dimensional non-zero Hilbert space H with spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$. Using the spectral resolution of T Prove the following statements:-

- (i) T is self-adjoint \Leftrightarrow each λ_i is real.
- (ii) T is Positive $\Leftrightarrow \lambda_i \geq 0$ for each i .
- (iii) T is unitary $\Leftrightarrow |\lambda_i| = 1$ for each i .

M.V. 94 Q.No. 98, 99 Let T be a normal operator on a finite-dimensional Hilbert space having the spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ then Prove that

- (i) T is self-adjoint \Leftrightarrow each λ_i is real
- (ii) T is unitary $\Leftrightarrow |\lambda_i| = 1$ for each i .

Solnⁿ Let M_1, M_2, \dots, M_m be the eigenspaces of T corresponding to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ and P_1, P_2, \dots, P_m be the projections on M_1, M_2, \dots, M_m respectively then the normal operator T has a spectral resolution,

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m.$$

$$\therefore T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \dots + \bar{\lambda}_m P_m$$

(i) T is self-adjoint $\Leftrightarrow T = T^*$

$$\Leftrightarrow \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \dots + \bar{\lambda}_m P_m$$

$$\Leftrightarrow (\lambda_1 - \bar{\lambda}_1) P_1 + (\lambda_2 - \bar{\lambda}_2) P_2 + \dots + (\lambda_m - \bar{\lambda}_m) P_m = 0$$

Clearly, if each λ_i is real then $\lambda_i = \bar{\lambda}_i$ for each i . Hence (1) is satisfied, therefore, T is self-adjoint.

Now, since P_i 's are pairwise orthogonal, if T is self-adjoint then since,

$$(\lambda_1 - \bar{\lambda}_1)P_1 + (\lambda_2 - \bar{\lambda}_2)P_2 + \dots + (\lambda_m - \bar{\lambda}_m)P_m = 0,$$

we have

$$P_i \{ (\lambda_1 - \bar{\lambda}_1)P_1 + (\lambda_2 - \bar{\lambda}_2)P_2 + \dots + (\lambda_i - \bar{\lambda}_i)P_i + \dots + (\lambda_m - \bar{\lambda}_m)P_m \} \\ = P_i \cdot 0 = 0.$$

$$\therefore (\lambda_1 - \bar{\lambda}_1)P_i P_1 + (\lambda_2 - \bar{\lambda}_2)P_i P_2 + \dots + (\lambda_i - \bar{\lambda}_i)P_i + \dots \\ + (\lambda_m - \bar{\lambda}_m)P_i P_m = 0$$

But, $P_i P_j = 0$ for $i \neq j$. $\therefore P_i P_1 = P_i P_2 = \dots = P_i P_m = 0$.

$$\therefore (\lambda_i - \bar{\lambda}_i)P_i = 0 \therefore \lambda_i - \bar{\lambda}_i = 0 \text{ or } \lambda_i = \bar{\lambda}_i.$$

Therefore, each λ_i is real. Thus if T is self-adjoint, each λ_i is real. i.e. T is self-adjoint \Leftrightarrow each λ_i is real.

$$(ii) \text{ For } x \in H, (Tx, x) = \sum_{i=1}^m \lambda_i \|P_i x\|^2 \quad \text{--- (2)}$$

If T is a +ve operator then $(Tx, x) \geq 0 \forall x \in H$.

$$\therefore \text{ from (2) } \sum_{i=1}^m \lambda_i \|P_i x\|^2 \geq 0, \forall x \in H.$$

Now, for any fixed i , let $x \in \text{range of } P_i$. then $P_i x = x$ & $P_j x = 0$ for $j \neq i$.

$$\therefore \lambda_i \|x\|^2 \therefore \lambda_i \geq 0 \text{ for each } i.$$

Conversely, if each $\lambda_i \geq 0$, then each λ_i is real.

Hence by (i) T is self-adjoint.

Also $\|P_i x\|^2 \geq 0 \therefore (Tx, x) \geq 0 \therefore T$ is positive. i.e. T is positive $\Leftrightarrow \lambda_i \geq 0$ for each i .

$$(ii) T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m.$$

$$T \text{ unitary} \Leftrightarrow T T^* = T^* T = I.$$

$$\text{Now, } T^* T = (\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m) (\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m)^*.$$

$$= (\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m) (\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \dots + \bar{\lambda}_m P_m)$$

$$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \dots + |\lambda_m|^2 P_m.$$

if $|\lambda_i| = 1$ for each i , then $T T^* = P_1 + P_2 + \dots + P_m = I.$

Similarly $T^* T = P_1 + P_2 + \dots + P_m = I.$

Thus, $T T^* = T^* T = I$. Hence T is unitary.

Conversely, if T is unitary then $T T^* = T^* T = I.$

$$\text{So, } |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \dots + |\lambda_m|^2 P_m = I$$

$$\therefore P_i (|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \dots + |\lambda_i|^2 P_i^2 + \dots + |\lambda_m|^2 P_i P_m) = P_i.$$

$$\therefore |\lambda_i|^2 P_i = P_i \quad [\because P_i^2 = P_i]$$

$$(|\lambda_i|^2 - 1) P_i = 0 \text{ for each } i.$$

Since $P_i \neq 0$, so $|\lambda_i|^2 - 1 = 0$ or, $|\lambda_i| = 1$ for each i .

i.e. T is unitary $\Leftrightarrow |\lambda_i| = 1$ for each i .

M.U.
M.No. 92, 96

Q.No → Prove that the spectral resolution of a normal operator on a finite-dimensional Hilbert space is unique.

Proof:- We know that, if T is normal operator then T can be expressed as,

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m \quad \text{--- (1)}$$

Where P_i 's are pairwise orthogonal. This expression for T is known as spectral resolution of T . Here we prove that this expression T is unique except for notations and order of terms. Since P_i 's are pairwise orthogonal idempotent

$$T^2 = \sum_{i=1}^m \lambda_i^2 P_i + \cancel{2 \sum_{i < j} P_i P_j \lambda_i \lambda_j}$$

$$T^2 = \sum_{i=1}^m \lambda_i^2 P_i$$

and in general, if n is any +ve integer

$$T^n = \sum_{i=1}^m \lambda_i^n P_i \quad \text{--- (2)}$$

Let $T^0 = I$, the identity operator, since $I = \sum_{i=1}^m P_i$.

We see that (2) holds, also for $n=0$. Let

$p(z)$ be a Polynomial in the complex variable z with complex coefficients. Hence by using (2)

$$p(T) = \sum_{i=1}^m p(\lambda_i) P_i \quad \text{--- (3)}$$

If j is any +ve integer from 1 to m , we enquire a Polynomial $p_j(z)$ such that all the coefficients on the right hand side of (3) vanish except that of P_j , whose coefficient should be 1.

$$\text{Clearly, } p_j(z) = \frac{(z-\lambda_1)(z-\lambda_2)\dots(z-\lambda_{j-1})(z-\lambda_{j+1})\dots(z-\lambda_m)}{(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\dots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\dots(\lambda_j-\lambda_m)}.$$

Therefore, p_j 's a Polynomial such that

$$p_j(\lambda_i) = 0 \text{ if } i \neq j$$

Hence, (3) gives $P_j = p_j(T)$ — (4)

Therefore P_j is Spectral Polynomial in T with Complex Coefficient. Therefore the expression for P_j given in (4) has been obtained under the assumption as λ_i 's are distinct Complex numbers and P_i 's ~~are~~ as known zero pairwise orthogonal Projections, and

$$I = \sum_{i=1}^m P_i.$$

If possible, let there be another Spectral resolution for T given by,

$$T = \mu_1 Q_1 + \mu_2 Q_2 + \dots + \mu_k Q_k \text{ — (5)}$$

where μ_i 's distinct Complex numbers, Q_i 's are non-zero pairwise orthogonal Projection where $I = \sum_{i=1}^k Q_i$. We next show that

① & ⑤ are the same except for notations and order of term which will complete the proof.

Since, \mathcal{O}_i is non-zero, there exists a non-zero element x in the range of \mathcal{O}_i , where $\mathcal{O}_i x = x$. If $x = y + z$ is the decomposition of x and $j \neq i$, then $\mathcal{O}_j x = y$. Since \mathcal{O}_i is orthogonal to \mathcal{O}_j , $j \neq i$, $x \perp y$ from which derived the $y = 0 \therefore \mathcal{O}_j x = 0$. Therefore, ⑤ gives, $Tx = \mu_i x$.

Hence, μ_i is an eigenvalue of T . Now, if λ is an eigenvalue of T then $Tx = \lambda x$ for some non-zero x and,

$$\therefore Tx = \lambda Ix = \lambda \sum_{i=1}^k \mathcal{O}_i x = \sum_{i=1}^k \lambda \mathcal{O}_i x$$

and $Tx = \sum_{i=1}^k \mu_i \mathcal{O}_i x$ from ⑤

$$\therefore \sum_{i=1}^k (\lambda - \mu_i) \mathcal{O}_i x = 0 \quad \text{--- ⑥}$$

Now, \mathcal{O}_i is orthogonal to \mathcal{O}_j for $i \neq j$ and they are projections therefore $\mathcal{O}_i x \perp \mathcal{O}_j x$ for $i \neq j$.

$$\therefore Ix = \sum_{i=1}^k \mathcal{O}_i x = 0 \quad [\text{Since } x \neq 0]$$

~~Therefore~~, therefore, that $\mathcal{O}_l(x) \neq 0$ where $1 \leq l \leq k$.

Taking inner product of $\mathcal{O}_l(x)$ with ⑥, we obtain

$\lambda - \mu_l = 0$ i.e. $\lambda = \mu_l$. Thus we see that the set

of μ_i 's is equal to the set of λ_i 's and

therefore ⑤ may be rewritten in the form

$$T = \lambda_1 \mathcal{O}_1 + \lambda_2 \mathcal{O}_2 + \dots + \lambda_m \mathcal{O}_m.$$

The relation (4), applied to this,

$$\therefore Q_j = P_j(T)$$

For every j & Q_j 's are equal to the P_j 's, and this proves the uniqueness of the spectral resolution of T .